Small-q_T factorization and its use for higher-order calculations in QCD

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In collaboration with

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Outline

1. The cross sections in QCD factorize in the limit of small transverse momentum, q_T, of the produced object

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2. The ingredients which appears in the factorization formula are known as the hard, the soft and the beam functions
3. I shall present the complete result for the NNLO soft function for top pair production and report on progress towards the $\mathrm{N}^3\mathrm{LO}$ beam functions

Big picture

- Each collision at the LHC involves interactions of quarks and gluons
  $\hookrightarrow$ **Understanding of strong interactions is critical to fully exploit potential of the LHC at the new energy frontier**
- Stringent limits on BSM have been set. So far, no new physics
  $\hookrightarrow$ **This calls for even more precise theoretical predictions**

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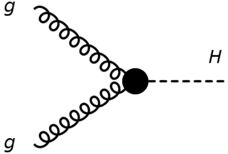
Predictions in perturbative QCD

- In the region where the strong coupling $\alpha_s \ll 1$, fixed-order perturbative expansions is expected to work well

$$\sigma = \underbrace{\sigma_0}_{\text{LO}} + \underbrace{\alpha_s\sigma_1}_{\text{NLO}} + \underbrace{\alpha_s^2\sigma_2}_{\text{NNLO}} + \underbrace{\alpha_s^3\sigma_3}_{\text{N}^3\text{LO}} + \cdots$$

Anatomy of perturbative QCD calculations

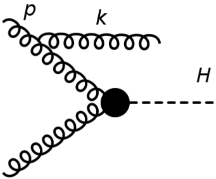
Leading Order (LO)



Anatomy of perturbative QCD calculations

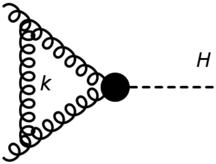
Next-to-Leading Order (NLO)

Real $\int d^4 k$



$\Rightarrow$ divergent in the limits $k \to 0$ or $k \parallel p$

Virtual $\int d^{4-2\epsilon} k$



$$= \quad \frac{1}{\epsilon^2} + \frac{1}{\epsilon} + \text{finite}$$

Anatomy of perturbative QCD calculations

$$\sigma_{\text{NLO}} = R + V$$

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$\hookrightarrow$ Divergences of R and V have to cancel

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How to carry out this cancellation in practice, given that *R* is integrated in 4 while *V* in *d* dimensions?

Anatomy of perturbative QCD calculations

- Subtraction

$$\boxed{d = 4 - 2\epsilon}$$

$$\sigma_{\text{NLO}} = \lim_{\epsilon \to 0} \left\{ \int d^d k \, R + \int d^d k \, V \right\}$$

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$S \simeq R$ in soft/collinear limit but simpler, hence integrable analytically in d dimensions

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$$\sigma_{\text{NLO}} = \int d^4 k\, (R + V)\left\{\Theta(\chi_{\text{cut}} - \chi(k)) + \Theta(\chi(k) - \chi_{\text{cut}})\right\}$$

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$$= \underbrace{\int d^4 k\,(R + V)\,\Theta(\chi_{\text{cut}} - \chi(k))}_{\text{unresolved}} + \underbrace{\int d^4 k\, R\,\Theta(\chi(k) - \chi_{\text{cut}})}_{\text{resolved}}$$

The q_T slicing method

[Catani, Grazzini ‘07, ‘15]

$$p + p \to F\,(q_T) + X$$

$$\sigma^{F}_{\mathrm{N}^m\mathrm{LO}} = \int_0^{q_{T,\mathrm{cut}}} dq_T\, \frac{d\sigma^{F}_{\mathrm{N}^m\mathrm{LO}}}{dq_T} + \int_{q_{T,\mathrm{cut}}}^{\infty} dq_T\, \frac{d\sigma^{F}_{\mathrm{N}^m\mathrm{LO}}}{dq_T}$$

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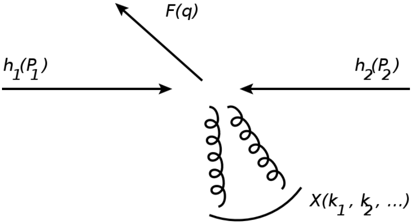
enough to know in
small-q_T approximation

known

Factorization



where $F = H, Z, W, ZZ, WW, t\bar{t}, \ldots$

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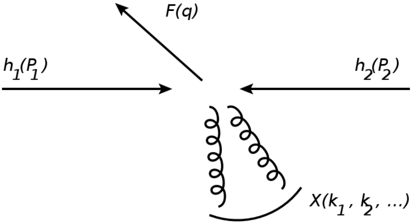

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- $q^2 \sim q_T^2 \gg \Lambda_{\rm QCD}$ collinear factorization

$$\frac{d\sigma_F}{d\Phi} = \phi_1 \otimes \phi_2 \otimes C + \mathcal{O}\left(\frac{1}{q^2}\right)$$

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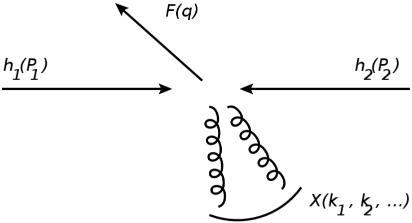

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- $q^2 \gg q_T^2 > \Lambda_{\rm QCD}$ small-q_T factorization

$$\frac{d\sigma_F}{d\Phi} = \mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{H} \otimes \mathcal{S} + \mathcal{O}\left(\frac{q_T^2}{q^2}\right)$$

All those functions

To get the cross section at $\mathrm{N}^m\mathrm{LO}$, we need to know all those functions at $\mathrm{N}^m\mathrm{LO}$

$$\frac{d\sigma_F^{\mathrm{N}^m\mathrm{LO}}}{d\Phi} = \mathcal{B}_1^{\mathrm{N}^m\mathrm{LO}} \otimes \mathcal{B}_2^{\mathrm{N}^m\mathrm{LO}} \otimes \mathcal{H}^{\mathrm{N}^m\mathrm{LO}} \otimes \mathcal{S}^{\mathrm{N}^m\mathrm{LO}}$$

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$\mathcal{B}$ - beam function - radiation collinear to the beam, process-independent, known up to NNLO

$\mathcal{H}$ - hard function - virtual corrections, process-dependent

$\mathcal{S}$ - soft function - soft, real radiation, process-dependent

Today, I will focus on Sand B.

Renormalization

separately divergent

$$\frac{d\sigma_F}{d\Phi} = \mathcal{B}_1^{(\mathsf{bare})} \otimes \mathcal{B}_2^{(\mathsf{bare})} \otimes \mathsf{Tr}\left[\mathcal{H}^{(\mathsf{bare})} \otimes \mathcal{S}^{(\mathsf{bare})}\right]$$

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&= \mathcal{B}_1(\mu) \otimes \mathcal{B}_2(\mu) \otimes \text{Tr}\left[\mathcal{H}(\mu) \otimes \mathcal{S}(\mu)\right]
\end{aligned}$$

separately finite

$$\frac{d}{d\mu}\frac{d\sigma_F}{d\Phi} = 0 \quad \rightarrow \quad \text{Renormalization Group Equations for } \mathcal{B}, \mathcal{H} \text{ and } \mathcal{S}$$

Soft Collinear Effective Theory (SCET)

$$\text{SCET} \simeq \text{QCD}\Big|_{\text{IR limit}}$$

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The new fields decouple in the Lagrangian

$$\mathcal{L}_{\mathrm{SCET}} = \mathcal{L}_c + \mathcal{L}_{\bar{c}} + \mathcal{L}_s$$

- The separation of fields in the Lagrangian into collinear, anti-collinear and soft sectors, facilitates proofs of factorization theorems

Small-q_T factorization in SCET

Gluons' momenta in light-cone coordinates

$$k_i^\mu = \left(k_i^+, k_i^-, \boldsymbol{k}_i^\perp\right) \qquad \text{where} \qquad k^\pm = k^0 \pm k^3$$

Expansion parameter

$$\lambda = \sqrt{\frac{q_T^2}{q^2}} \ll 1$$

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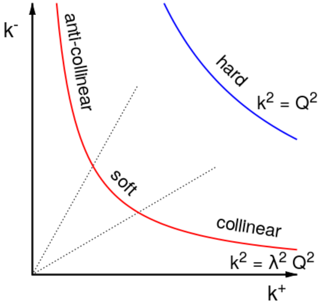
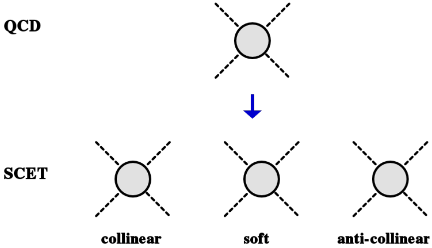
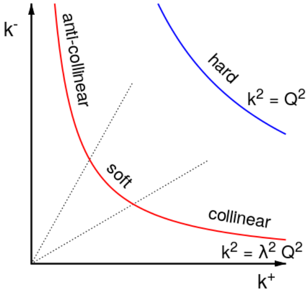
Regions

collinear	$k_i^\mu \sim (1, \lambda^2, \lambda)\, Q^2$	$\mathcal{B}_1$
anti-collinear	$k_i^\mu \sim (\lambda^2, 1, \lambda)\, Q^2$	$\mathcal{B}_2$
hard	$k_i^\mu \sim (1, 1, 1)\, Q^2$	$\mathcal{H}$
soft	$k_i^\mu \sim (\lambda, \lambda, \lambda)\, Q^2$	$\mathcal{S}$



Rapidity divergences and analytic regulator





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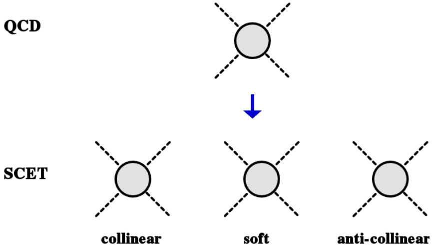
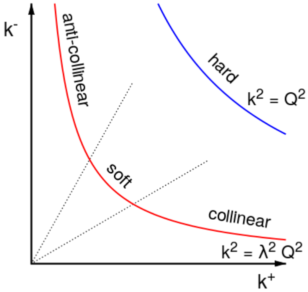




Modification of the measure [Becher, Bell '12]

$$\int d^d k \, \delta^+(k^2) \to \int d^d k \left(\frac{\nu}{k_+}\right)^{\alpha} \delta^+(k^2)$$

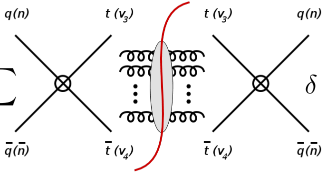
- The regulator is necessary at intermediate steps of the calculation.
- Rapidity divergences do not appear in QCD, hence, the complete SCET result has to stay finite in the limit $\alpha \to 0$.

NNLO soft function for top pair production

Soft function

- Represents corrections coming from exchanges of real, soft gluons, whose transverse momenta sum up to a fixed value q_T

$$S_{\text{bare}}(q_T, \beta_t, \theta) \propto \sum$$



$$\delta\left(q_T - \left|\sum_i k_{i\perp}\right|\right) \prod_i \delta^+(k_i^2)$$

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$$\boxed{\beta_t = \sqrt{1 - \frac{4\,m_t^2}{M^2}}}$$

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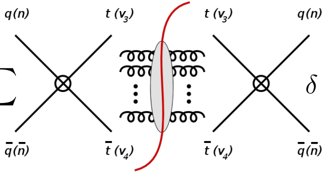

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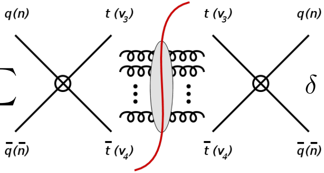

$$\delta\left(q_T - |\textstyle\sum_i k_{i\perp}|\right) \prod_i \delta^+(k_i^2)$$

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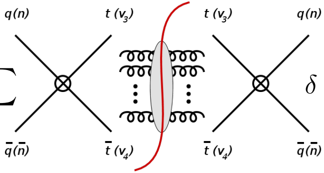

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$$\boldsymbol{S}_{i\bar{i}} = \sum_{n=0}^{\infty} \boldsymbol{S}_{i\bar{i}}^{(n)} \left(\frac{\alpha_s}{4\pi}\right)^n \qquad \boldsymbol{S}_{i\bar{i}}^{(n)} = \sum_{\{j\}} \boldsymbol{w}_{\{j\}}^{i\bar{i}} I_{\{j\}}$$

colour matrices ⮥ ⮤ phase space integrals

Renormalization

- RG equation for the soft function

$$\frac{d}{d\ln\mu}\boldsymbol{S}_{i\bar{i}}(\mu) = -\gamma^{s\dagger}_{i\bar{i}}\,\boldsymbol{S}_{i\bar{i}}(\mu) - \boldsymbol{S}_{i\bar{i}}(\mu)\,\gamma^{s}_{i\bar{i}}$$

- Soft anomalous dimension

$$\gamma^{s} = -\boldsymbol{Z}_s^{-1}\frac{d\boldsymbol{Z}_s}{d\ln\mu}$$

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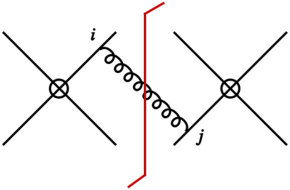
Specifically, at the order α_s^2, we get

$$\underbrace{\boldsymbol{S}^{(2)}}_{\text{finite part only}} = \overbrace{\boldsymbol{Z}_s^{\dagger(2)}\boldsymbol{S}^{(0)}_{\text{bare}} + \boldsymbol{S}^{(0)}_{\text{bare}}\boldsymbol{Z}_s^{(2)} + \boldsymbol{Z}_s^{\dagger(1)}\boldsymbol{S}^{(0)}_{\text{bare}}\boldsymbol{Z}_s^{(1)}}^{\text{pole part only}}$$

$$+ \underbrace{\boldsymbol{Z}_s^{\dagger(1)}\boldsymbol{S}^{(1)}_{\text{bare}} + \boldsymbol{S}^{(1)}_{\text{bare}}\boldsymbol{Z}_s^{(1)} + \boldsymbol{S}^{(2)}_{\text{bare}} - \frac{\beta_0}{\epsilon}\,\boldsymbol{S}^{(1)}_{\text{bare}}}_{\text{finite + pole part}}$$

Soft function at NLO



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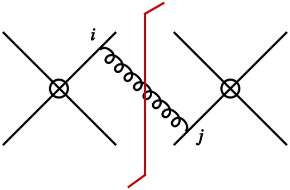


- Known in analytic form
  [Li, Li, Shao, Yan, Zhu '13; Catani, Grazzini, Torre '13]

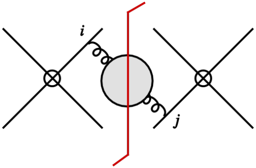
$$\boxed{L_\perp = \ln \frac{x_T^2 \mu^2}{4e^{-2\gamma_E}}}$$

$$\begin{aligned}
\boldsymbol{S}_{i\bar{i}}^{(1)} &= 4L_\perp \left(2\boldsymbol{w}_{i\bar{i}}^{13} \ln \frac{-t_1}{m_t M} + 2\boldsymbol{w}_{i\bar{i}}^{23} \ln \frac{-u_1}{m_t M} + \boldsymbol{w}_{i\bar{i}}^{33} \right) \\
&- 4\left(\boldsymbol{w}_{i\bar{i}}^{13} + \boldsymbol{w}_{i\bar{i}}^{23}\right) \mathrm{Li}_2\left(1 - \frac{t_1 u_1}{m_t^2 M^2}\right) + 4\boldsymbol{w}_{i\bar{i}}^{33} \ln \frac{t_1 u_1}{m_t^2 M^2} \\
&- 2\boldsymbol{w}_{i\bar{i}}^{34} \frac{1+\beta_t^2}{\beta_t} \left[L_\perp \ln x_s - \mathrm{Li}_2\left(-x_s \,\mathrm{tg}^2 \frac{\theta}{2}\right) + \mathrm{Li}_2\left(-\frac{1}{x_s} \,\mathrm{tg}^2 \frac{\theta}{2}\right) \right. \\
&\left. + 4 \ln x_s \ln \cos \frac{\theta}{2} \right] + \mathcal{O}(\epsilon)
\end{aligned}$$

Soft function at NNLO

Three distinct groups of diagrams:

- Bubble



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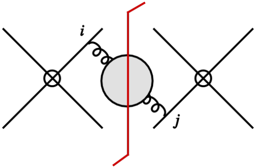
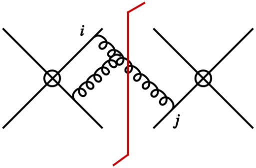


- Single-cut



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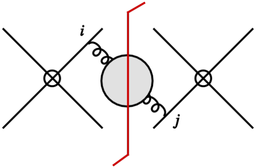

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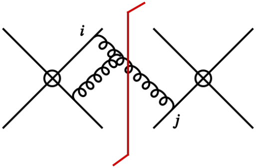

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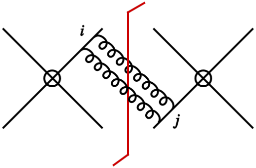

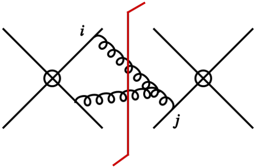

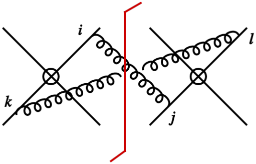

+ . . .

Soft function at NNLO

Three distinct groups of diagrams:

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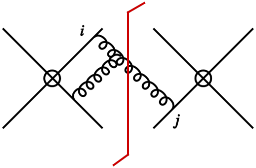
DIFFERENTIAL EQUATIONS

- Single-cut



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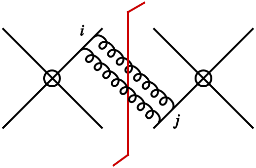

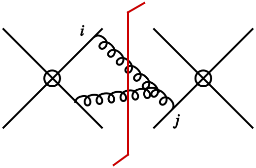

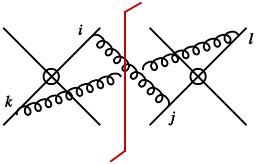

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Soft function at NNLO

Three distinct groups of diagrams:

- Bubble
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DIFFERENTIAL EQUATIONS

DIRECT INTEGRATION

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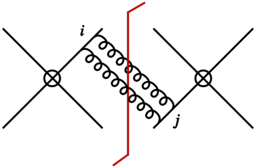

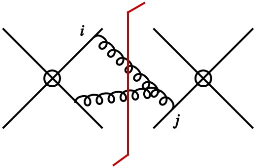

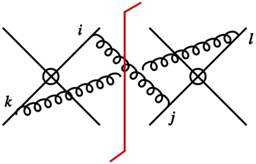

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SECTOR DECOMPOSITION

Double-cut NNLO integrals

Example:

$$\tilde{I}_{3gv,ij} = \int \frac{d^d k_1 \, d^d k_2 \; \delta^+(k_1^2)\, \delta^+(k_2^2)\, \delta((k_1+k_2)_T^2 - q_T^2)}{(n\cdot k_1)^\alpha \, (n\cdot k_2)^\alpha \, (n_i \cdot k_1)\, (n_j \cdot (k_1+k_2))\, (k_1+k_2)^2}$$

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- divergent in the limits $\epsilon \to 0$ and $\alpha \to 0$
- a range of overlapping singularities
- complication introduced by $\delta((k_1+k_2)_T^2 - q_T^2)$ which additionally couples gluon's momenta

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- complication introduced by $\delta((k_1+k_2)_T^2 - q_T^2)$ which additionally couples gluon's momenta

To disentangle overlapping singularities and calculate regularized integrals we use the method of sector decomposition [Binoth, Heinrich, '00; Borowka, Heinrich, Jahn, Jones, Kerner, Schlenk, Zirke '17].

Sector decomposition

$$\int_0^1 dx\, dy\, \frac{\mathcal{W}(x,y)}{(x+y)^{2+\epsilon}}$$

Sector decomposition

$$\int_0^1 dx\, dy \frac{\mathcal{W}(x,y)}{(x+y)^{2+\epsilon}} = \int_0^1 dx\, dy \frac{\mathcal{W}(x,y)}{(x+y)^{2+\epsilon}} \left[\overbrace{\Theta(x-y)}^{(1)} + \overbrace{\Theta(y-x)}^{(2)} \right]$$

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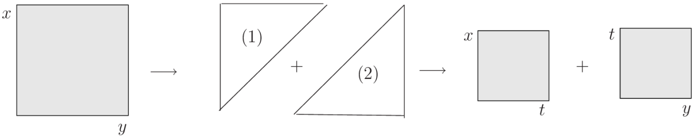

Sector decomposition

In general, each integral can be expressed as

$$\mathcal{I} = \sum_{i \,\in\, \text{sectors}} \int_0^1 \frac{dx_1}{x_1^{1+a_1\epsilon}} \frac{dx_2}{x_2^{1+a_2\epsilon}} \cdots \frac{dx_n}{x_n^{1+a_n\epsilon}} \mathcal{W}_i(x_1, x_2, \ldots, x_n)$$

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After the above procedure is performed, all divergences become explicit and are turned in to ϵ poles

$$\mathcal{I}_i = \sum_n \underbrace{\left(\int \mathcal{W}_{in}\right)}_{\text{finite}} \times \epsilon^n$$

Sector decomposition

Two types of singularities

- Endpoint, *e.g.* soft:

$$\left(k_1^+, k_1^-, k_1^\perp\right) \to 0$$

Sector decomposition

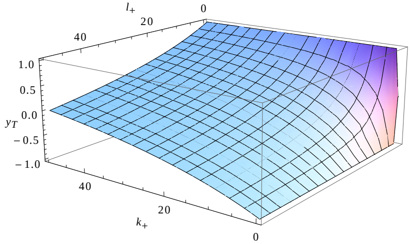
Two types of singularities

- Endpoint, *e.g.* soft:

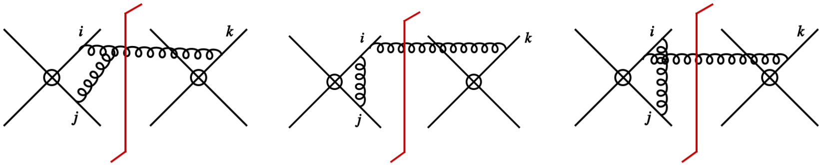
$$\left(k_1^+, k_1^-, k_1^\perp\right) \to 0$$

- Manifold, *e.g.* collinear

$$k_1 \cdot k_2 \to 0$$



Single-cut (real-virtual)



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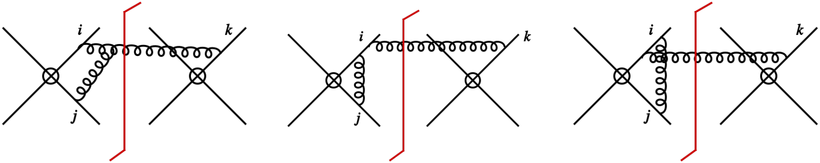

$$S^{(2)}_{\text{1-cut}} = \sum_{ijk} \int d^d l \, \frac{\delta^+(l^2)\, \delta(l_T - q_T)}{l_+^\alpha \, n_k \cdot l} \, n_k^\mu \, T_k^a \, J^\mu_{ij,a}(l)$$

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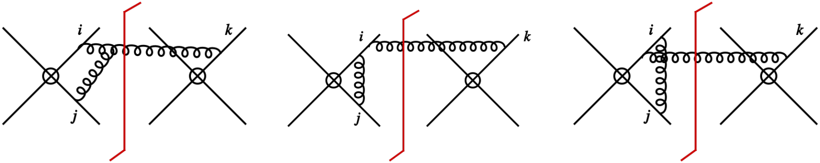

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- The soft current $J^\mu_{ij,a}(l)$ is known up to NLO [Catani, Grazzini '00; Bierenbaum, Czakon, Mitov '12; Czakon, Mitov '18].

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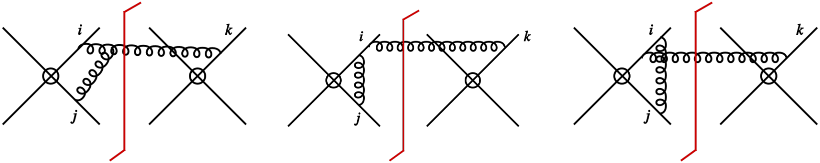

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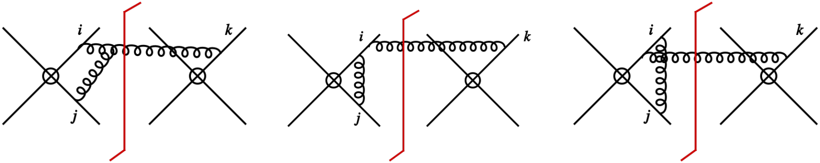

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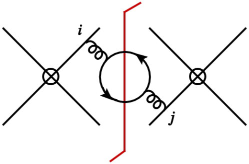
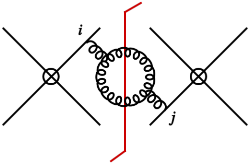
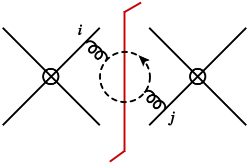
- $S^{(2)}_{\text{1-cut}}$ can be obtained by a relatively simple integration over l^μ.
- Single-cut piece of the soft function exhibits both real and imaginary part. The latter when $i \neq j \neq k$, the former, otherwise.

Bubble







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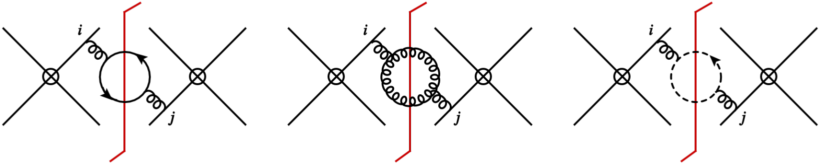

- Solvable analytically: direct cross check of our sector decomposition-based implementation
- Non-trivial tensor structure → challenging numerators
- Laboratory to stress-test sector decomposition-based methodology
- Comparable with n_f part of Renormalization Group prediction

Complete Soft Function at NNLO: structure of the result

- In momentum space

$$S^{(2,\text{bare})}(q_T,\beta_t,\theta)=\frac{1}{q_T^p}\left[S^{(2)}_{\text{bubble}}(\beta_t,\theta,\epsilon)+S^{(2)}_{\text{1-cut}}(\beta_t,\theta,\epsilon)+S^{(2)}_{\text{2-cut}}(\beta_t,\theta,\epsilon)\right]$$

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Fourier Transform

- In position space

$$S^{(2,\text{bare})}(L_\perp,\beta_t,\theta)=\left[\frac{1}{\epsilon}+L_\perp+L_\perp^2+\dots\right]$$

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$\hookrightarrow$ Momentum-space soft function has to be calculated up to order ϵ.

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$$= \frac{1}{\epsilon^2}S^{(2,-2)}(L_\perp) + \frac{1}{\epsilon}S^{(2,-1)}(L_\perp) + S^{(2,0)}(L_\perp)$$

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can be cross-checked against RG; fixes all $L_\perp$-dependent terms in $S^{(2,0)}(L_\perp)$

Complete Soft Function at NNLO: structure of the result

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\begin{aligned}
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- The only term that has to be obtained through direct calculation is the $L_\perp$-independent part of $S^{(2,0)}(L_\perp)$.
- However, we calculate all terms and use the redundant ones for cross checks against Renormalization Group prediction.

Vanishing of higher order poles

Even though the NNLO Soft Function exhibits at most $\frac{1}{\epsilon^2}$ singularity, higher order poles appear in individual contributions.

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Even though the NNLO Soft Function exhibits at most $\frac{1}{\epsilon^2}$ singularity, higher order poles appear in individual contributions.

- All α poles, including $\frac{\epsilon}{\alpha}$, as well as $\frac{1}{\epsilon^4}$ pole cancel within each colour structure, for example

$$\frac{1}{\epsilon^4}\begin{pmatrix} 0.00009\,N_c^{-1} - 0.00009\,N_c & -0.00002\,N_c^2 - 0.00009\,N_c^{-2} + 0.0001 \\ -0.00002\,N_c^2 - 0.00009\,N_c^{-2} + 0.0001 & 0.00008\,N_c^3 - 0.00006\,N_c + 0.00007\,N_c^{-3} - 0.00009\,N_c^{-1} \end{pmatrix}$$

Vanishing of higher order poles

Even though the NNLO Soft Function exhibits at most $\frac{1}{\epsilon^2}$ singularity, higher order poles appear in individual contributions.

- All α poles, including $\frac{\epsilon}{\alpha}$, as well as $\frac{1}{\epsilon^4}$ pole cancel within each colour structure, for example

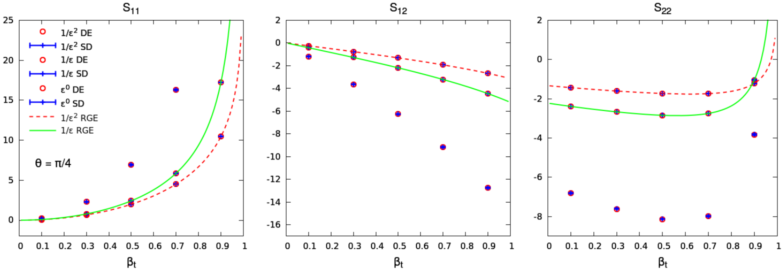
$$\frac{1}{\epsilon^4}\begin{pmatrix} 0.00009\,N_c^{-1} - 0.00009\,N_c & -0.00002\,N_c^2 - 0.00009\,N_c^{-2} + 0.0001 \\ -0.00002\,N_c^2 - 0.00009\,N_c^{-2} + 0.0001 & 0.00008\,N_c^3 - 0.00006\,N_c + 0.00007\,N_c^{-3} - 0.00009\,N_c^{-1} \end{pmatrix}$$

- $\frac{1}{\epsilon^3}$ pole cancels between 1-cut and 2-cut contributions

$$\frac{1}{\epsilon^3}\begin{pmatrix} 0.0004\,N_c^3 - 0.0007\,N_c + 0.0004\,N_c^{-1} & 0.0004\,N_c^2 - 0.0004\,N_c^{-2} - 7.\times 10^{-6} \\ 0.0004\,N_c^2 - 0.0004\,N_c^{-2} - 7.\times 10^{-6} & -0.0004\,N_c^3 - 0.00001\,N_c + 0.0003\,N_c^{-3} + 0.0002\,N_c^{-1} \end{pmatrix}$$

[†] We used $\beta_t = 0.4$, $\theta = 0.5$.

Quark bubble contribution ($q\bar{q}$ channel)

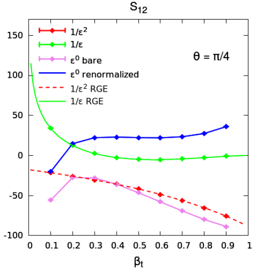


Validation of the framework

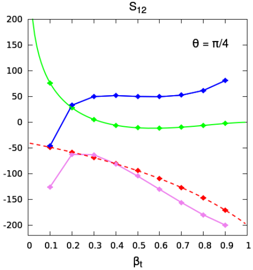
- Perfect agreement of the quark bubble results obtained from *differential equations* and *sector decomposition* for all terms in ϵ expansion
- Reproduction of the n_f part of the Renormalization Group result

Imaginary part

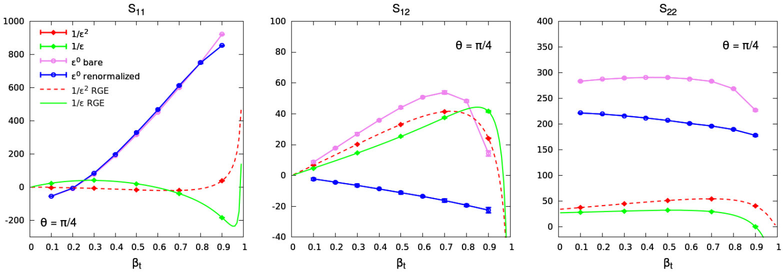
($q\bar{q}$ channel)



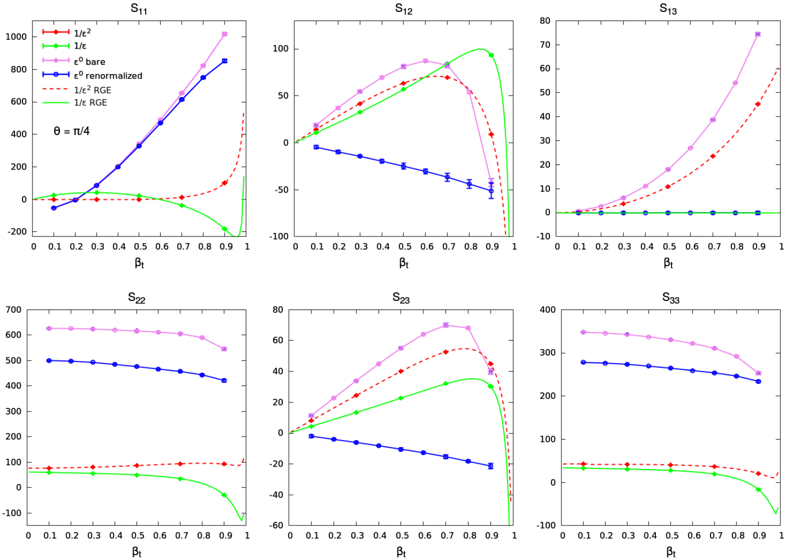
(gg channel)



Real part ($q\bar{q}$ channel)



Real part (gg channel)



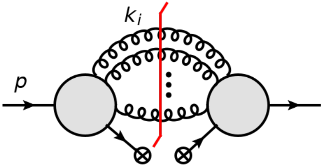
N^{3}LO beam function

(work in progress)

The beam function

- Represents corrections coming from emissions of real, collinear gluons, whose transverse momenta sum up to a fixed value q_T and whose longitudinal component along p sums up to $1-z$

$B_{\text{bare}}(q_T, z) \propto \sum$



$$\times\, \delta\left(q_T - \left|\textstyle\sum_i k_{i\perp}\right|\right) \textstyle\prod_i \delta^+(k_i^2)\, \delta\left(\bar{n}\cdot \textstyle\sum k_i - (1-z)\,\bar{n}\cdot p\right)$$

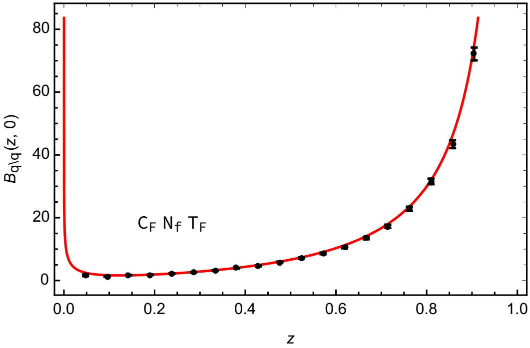
$$p = \frac{\bar{n}\cdot p}{2} n$$

$$n^2 = \bar{n}^2 = 0$$

$$n\cdot\bar{n} = 2$$

NNLO beam function

- Known analytically [Gehrmann, Lübbert, Yang '12, '14].
- We checked that our method reproduces that result



N^{3}LO propagators

light-cone

$n \cdot l_1$

$n \cdot l_2$

$n \cdot l_3$

$\bar{n} \cdot l_1$

$\bar{n} \cdot l_2$

$\bar{n} \cdot l_3$

$n \cdot l_1 + n \cdot l_2$

$n \cdot l_1 + n \cdot l_3$

$n \cdot l_2 + n \cdot l_3$

$\bar{n} \cdot l_1 + \bar{n} \cdot l_2$

$\bar{n} \cdot l_1 + \bar{n} \cdot l_3$

$\bar{n} \cdot l_2 + \bar{n} \cdot l_3$

internal only

$l_1 \cdot l_2$

$l_1 \cdot l_3$

$l_2 \cdot l_3$

$l_1 \cdot l_2 + l_1 \cdot l_3 + l_2 \cdot l_3$

internal+external

$p_- \, n \cdot l_1$

$p_- \, n \cdot l_2$

$p_- \, n \cdot l_3$

$l_1 \cdot l_2 - p_- \, n \cdot l_1 - p_- \, n \cdot l_2$

$l_1 \cdot l_3 - p_- \, n \cdot l_1 - p_- \, n \cdot l_3$

$l_2 \cdot l_3 - p_- \, n \cdot l_2 - p_- \, n \cdot l_3$

The way to go

The beam function

$$B_{\text{bare}}(z, q_T) = \sum_i \mathcal{I}_i \,,$$

can be calculated if each integral is represented as

$$\mathcal{I}_i = \sum_{j \,\in\, \text{sectors}} \int_0^1 \frac{dx_1}{x_1^{1+a_1\epsilon}} \frac{dx_2}{x_2^{1+a_2\epsilon}} \frac{dx_3}{x_3^{1+a_3\epsilon}} \frac{dx_4}{x_4^{1+a_4\epsilon}} \, dx_5 \cdots dx_9 \, \mathcal{W}_j(x_1, x_2, \ldots, x_9) \,.$$

The way to go

The beam function

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$$\mathcal{I}_i = \sum_{j \,\in\, \text{sectors}} \int_0^1 \frac{dx_1}{x_1^{1+a_1\epsilon}} \frac{dx_2}{x_2^{1+a_2\epsilon}} \frac{dx_3}{x_3^{1+a_3\epsilon}} \frac{dx_4}{x_4^{1+a_4\epsilon}} \, dx_5 \cdots dx_9 \, \mathcal{W}_j(x_1, x_2, \ldots, x_9) \, .$$

Then we can use

$$\frac{1}{x_i^{1+a_i\epsilon}} = -\frac{1}{a_i\epsilon}\delta(x_i) + \sum_{n=0}^{\infty} \frac{a_i^n \epsilon^n}{n!} \left[\frac{\log^n(x_i)}{x_i} \right]_+ \, .$$

N^{3}LO propagators

The first problem: It is impossible to parameterize the momenta such that all scalar products look simple simultaneously.

N^{3}LO propagators

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Example

$$n = [1,0,0,0,1] \qquad \bar{n} = [1,0,0,0,-1] \quad l_1 = \left[\frac{l_{1-}^2 + l_{1T}^2}{2\,l_{1-}}, 0, 0, 0, \frac{l_{1-}^2 - l_{1T}^2}{2\,l_{1-}}\right]$$

$$l_3 = \left[\frac{l_{3-}^2 + l_{3T}^2}{2\,l_{3-}},\ 0,\ l_{3T}\sin\chi_1,\ l_{3T}\cos\chi_1,\ \frac{l_{3-}^2 - l_{3T}^2}{2\,l_{3-}}\right]$$

$$l_2 = \left[\frac{l_{2-}^2 + l_{2+}^2}{2\,l_{2-}^2},\ l_{2T}\sin\phi_1\sin\phi_2,\ l_{2T}\cos\phi_2\sin\phi_1,\ l_{2T}\cos\phi_1,\ \frac{l_{2-}^2 - l_{2+}^2}{2\,l_{2-}}\right]$$

N^3LO propagators

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$$\bar{n}\cdot l_1 = l_{1-} \quad \bar{n}\cdot l_2 = l_{2-} \quad \bar{n}\cdot l_3 = l_{3-}$$

$$l_1\cdot l_2 = \frac{l_{1T}^2 l_{2-}}{2\, l_{1-}} + \frac{l_{2T}^2 l_{1-}}{2\, l_{2-}} - l_{1T} l_{2T}\cos\phi_1$$

N^{3}LO propagators

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$$l_2 = \left[\frac{l_{2-}^2 + l_{2+}^2}{2\,l_{2-}^2},\ l_{2T}\sin\phi_1\sin\phi_2,\ l_{2T}\cos\phi_2\sin\phi_1,\ l_{2T}\cos\phi_1,\ \frac{l_{2-}^2 - l_{2+}^2}{2\,l_{2-}}\right]$$

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$$l_1\cdot l_2 = \frac{l_{1T}^2 l_{2-}}{2\,l_{1-}} + \frac{l_{2T}^2 l_{1-}}{2\,l_{2-}} - l_{1T}l_{2T}\cos\phi_1 \quad \Rightarrow \quad \phi_1 = 0 \ \& \ \frac{l_{1T}}{l_{1-}} = \frac{l_{2T}}{l_{2-}}$$

N^3LO propagators

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$$l_2 = \left[\frac{l_{2-}^2 + l_{2+}^2}{2\,l_{2-}^2},\ l_{2T}\sin\phi_1\sin\phi_2,\ l_{2T}\cos\phi_2\sin\phi_1,\ l_{2T}\cos\phi_1,\ \frac{l_{2-}^2 - l_{2+}^2}{2\,l_{2-}}\right]$$

$$\bar{n}\cdot l_1 = l_{1-} \quad \bar{n}\cdot l_2 = l_{2-} \quad \bar{n}\cdot l_3 = l_{3-}$$

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$$l_2\cdot l_3 = \frac{l_{2T}^2 l_{3-}}{2\,l_{2-}} + \frac{l_{3T}^2 l_{2-}}{2\,l_{3-}} - l_{2T} l_{3T}\cos\chi_1\cos\phi_1 - l_{2T} l_{3T}\cos\phi_2\sin\chi_1\sin\phi_1$$

Step 1: selector functions

7 triple collinear
$(l_1 \cdot l_2)\,(n \cdot l_1)\,(n \cdot l_2)$
$(l_1 \cdot l_3)\,(n \cdot l_1)\,(n \cdot l_3)$
$(l_2 \cdot l_3)\,(n \cdot l_2)\,(n \cdot l_3)$
$(l_1 \cdot l_2)\,(\bar{n} \cdot l_1)\,(\bar{n} \cdot l_2)$
$(l_1 \cdot l_3)\,(\bar{n} \cdot l_1)\,(\bar{n} \cdot l_3)$
$(l_2 \cdot l_3)\,(\bar{n} \cdot l_2)\,(\bar{n} \cdot l_3)$
$(l_1 \cdot l_2)\,(l_1 \cdot l_3)\,(l_2 \cdot l_3)$

12 double collinear	
$(n \cdot l_1)\,(\bar{n} \cdot l_2)$	$(l_1 \cdot l_3)\,(n \cdot l_2)$
$(n \cdot l_1)\,(\bar{n} \cdot l_3)$	$(l_2 \cdot l_3)\,(n \cdot l_1)$
$(n \cdot l_2)\,(\bar{n} \cdot l_3)$	$(l_1 \cdot l_2)\,(n \cdot l_3)$
$(\bar{n} \cdot l_1)\,(n \cdot l_2)$	$(l_1 \cdot l_3)\,(\bar{n} \cdot l_2)$
$(\bar{n} \cdot l_1)\,(n \cdot l_3)$	$(l_2 \cdot l_3)\,(\bar{n} \cdot l_1)$
$(\bar{n} \cdot l_2)\,(n \cdot l_3)$	$(l_1 \cdot l_2)\,(\bar{n} \cdot l_3)$

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$(l_1 \cdot l_3)\,(\bar{n} \cdot l_1)\,(\bar{n} \cdot l_3)$
$(l_2 \cdot l_3)\,(\bar{n} \cdot l_2)\,(\bar{n} \cdot l_3)$
$(l_1 \cdot l_2)\,(l_1 \cdot l_3)\,(l_2 \cdot l_3)$

12 double collinear	
$(n \cdot l_1)\,(\bar{n} \cdot l_2)$	$(l_1 \cdot l_3)\,(n \cdot l_2)$
$(n \cdot l_1)\,(\bar{n} \cdot l_3)$	$(l_2 \cdot l_3)\,(n \cdot l_1)$
$(n \cdot l_2)\,(\bar{n} \cdot l_3)$	$(l_1 \cdot l_2)\,(n \cdot l_3)$
$(\bar{n} \cdot l_1)\,(n \cdot l_2)$	$(l_1 \cdot l_3)\,(\bar{n} \cdot l_2)$
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$(\bar{n} \cdot l_2)\,(n \cdot l_3)$	$(l_1 \cdot l_2)\,(\bar{n} \cdot l_3)$

$$d_{1,2;1} = (l_1 \cdot l_2)\,(\bar{n} \cdot l_1)\,(\bar{n} \cdot l_2)\,,$$

$$S_{1,2;2} = \frac{1}{d_{1,2;1}\mathcal{D}}\,,$$

$$\mathcal{D} = \sum_{i,j,k} \frac{1}{d_{i,j;k}} + \sum_{i,j,k,l} \frac{1}{d_{i,j;k,l}}\,,$$

Step 1: selector functions

7 triple collinear
$(l_1 \cdot l_2)\,(n \cdot l_1)\,(n \cdot l_2)$
$(l_1 \cdot l_3)\,(n \cdot l_1)\,(n \cdot l_3)$
$(l_2 \cdot l_3)\,(n \cdot l_2)\,(n \cdot l_3)$
$(l_1 \cdot l_2)\,(\bar{n} \cdot l_1)\,(\bar{n} \cdot l_2)$
$(l_1 \cdot l_3)\,(\bar{n} \cdot l_1)\,(\bar{n} \cdot l_3)$
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$(n \cdot l_1)\,(\bar{n} \cdot l_3)$	$(l_2 \cdot l_3)\,(n \cdot l_1)$
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$(\bar{n} \cdot l_1)\,(n \cdot l_2)$	$(l_1 \cdot l_3)\,(\bar{n} \cdot l_2)$
$(\bar{n} \cdot l_1)\,(n \cdot l_3)$	$(l_2 \cdot l_3)\,(\bar{n} \cdot l_1)$
$(\bar{n} \cdot l_2)\,(n \cdot l_3)$	$(l_1 \cdot l_2)\,(\bar{n} \cdot l_3)$

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$$\mathcal{D} = \sum_{i,j,k} \frac{1}{d_{i,j;k}} + \sum_{i,j,k,l} \frac{1}{d_{i,j;k,l}}\,,$$

$$S_{1,2;2} = \frac{1}{1 + \dfrac{(l_1 \cdot l_2)\,(\bar{n} \cdot l_2)}{(l_1 \cdot l_3)\,(\bar{n} \cdot l_3)} + \dfrac{(l_1 \cdot l_2)\,(\bar{n} \cdot l_1)}{(l_1 \cdot l_3)} + \cdots}\,,$$

Step 2: sector decomposition

Let's focus on the sector $(l_1 \cdot l_2)\,(\bar{n} \cdot l_1)\,(\bar{n} \cdot l_2)$. All other singularities are suppressed by the corresponding selector functions.

In this sector, divergencies can be generated by the following propagators:

$$\bar{n} \cdot l_1$$

$$\bar{n} \cdot l_2$$

$$n \cdot l_1$$

$$n \cdot l_2$$

$$l_1 \cdot l_2$$

$$n \cdot l_1 + n \cdot l_2$$

$$\bar{n} \cdot l_1 + \bar{n} \cdot l_2$$

$$l_1 \cdot l_2 + l_1 \cdot l_3 + l_2 \cdot l_3$$

Step 2: sector decomposition

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In this sector, divergencies can be generated by the following propagators:

$$\begin{array}{rcl}
\bar{n} \cdot l_1 & \longrightarrow & l_{1-} \\
\bar{n} \cdot l_2 & \longrightarrow & l_{2-} \\
n \cdot l_1 & & \\
n \cdot l_2 & & \\
l_1 \cdot l_2 & \longrightarrow & \dfrac{l_{1T}^2 l_{2-}}{2 l_{1-}} + \dfrac{l_{2T}^2 l_{1-}}{2 l_{2-}} - l_{1T} l_{2T} \cos\phi_1 \\
n \cdot l_1 + n \cdot l_2 & & \\
\bar{n} \cdot l_1 + \bar{n} \cdot l_2 & \longrightarrow & l_{1-} + l_{2-} \\
l_1 \cdot l_2 + l_1 \cdot l_3 + l_2 \cdot l_3 & &
\end{array}$$

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n \cdot l_1 & & \\
n \cdot l_2 & & \\
l_1 \cdot l_2 & \longrightarrow & \dfrac{l_{1T}^2 l_{2-}}{2 l_{1-}} + \dfrac{l_{2T}^2 l_{1-}}{2 l_{2-}} - l_{1T} l_{2T} \cos\phi_1 \\
n \cdot l_1 + n \cdot l_2 & & \\
\bar{n} \cdot l_1 + \bar{n} \cdot l_2 & \longrightarrow & l_{1-} + l_{2-} \\
l_1 \cdot l_2 + l_1 \cdot l_3 + l_2 \cdot l_3 & &
\end{array}$$

Step 2: sector decomposition

The nonlinear transformation

$$\zeta = \frac{1}{2}\frac{(l_{1T}l_{2-} - l_{1-}l_{2T})^2\,(1+\cos\phi_1)}{l_{1T}^2 l_{2-}^2 + l_{1-}^2 l_{2T}^2 - 2l_{1-}l_{2-}l_{1T}l_{2T}\cos\phi_1}$$

Step 2: sector decomposition

The nonlinear transformation

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turns

$$l_1 \cdot l_2 = \frac{l_{1T}^2 l_{2-}}{2l_{1-}} + \frac{l_{2T}^2 l_{1-}}{2l_{2-}} - l_{1T}l_{2T}\cos\phi_1$$

Step 2: sector decomposition

The nonlinear transformation

$$\zeta = \frac{1}{2}\frac{(l_{1T}l_{2-} - l_{1-}l_{2T})^2\,(1+\cos\phi_1)}{l_{1T}^2 l_{2-}^2 + l_{1-}^2 l_{2T}^2 - 2l_{1-}l_{2-}l_{1T}l_{2T}\cos\phi_1}$$

turns

$$l_1\cdot l_2 = \frac{l_{1T}^2 l_{2-}}{2l_{1-}} + \frac{l_{2T}^2 l_{1-}}{2l_{2-}} - l_{1T}l_{2T}\cos\phi_1$$

into

$$l_1\cdot l_2 = \frac{(l_{1T}^2 l_{2-}^2 - l_{1-}^2 l_{2T}^2)^2}{2\,l_{1-}l_{2-}(l_{1T}^2 l_{2-}^2 + l_{1-}^2 l_{2T}^2 - 2\,l_{1-}l_{2-}l_{1T}l_{2T}(1-2\zeta))}$$

Step 2: sector decomposition

$$
\begin{aligned}
l_{1-} &= l_{1T}\, l_{1-} \\
l_{2-} &= l_{2T}\, l_{2-}
\end{aligned}
$$

$\boxed{l_{2-} \to l_{2-} l_{1-}}$ $\quad l_{1-} > l_{2-}$

$l_{1-} < l_{2-} \quad$ $\boxed{l_{1-} \to l_{1-} l_{2-}}$

Step 2: sector decomposition

$$\begin{aligned} l_{1-} &= l_{1T}\, l_{1-} \\ l_{2-} &= l_{2T}\, l_{2-} \end{aligned}$$

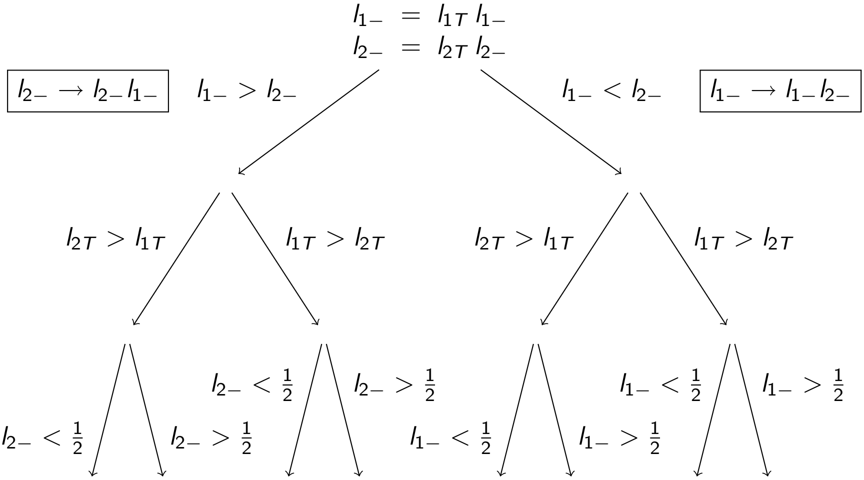
$\boxed{l_{2-} \to l_{2-} l_{1-}}$ $l_{1-} > l_{2-}$

$l_{1-} < l_{2-}$ $\boxed{l_{1-} \to l_{1-} l_{2-}}$

$$l_{1-} + l_{2-} \to l_{1-}(1 + l_{2-})$$

$$l_{1-} + l_{2-} \to l_{2-}(1 + l_{1-})$$

Step 2: sector decomposition



$$\begin{aligned} l_{1-} &= l_{1T}\, l_{1-} \\ l_{2-} &= l_{2T}\, l_{2-} \end{aligned}$$

$\boxed{l_{2-} \to l_{2-} l_{1-}}$ $l_{1-} > l_{2-}$

$l_{1-} < l_{2-}$ $\boxed{l_{1-} \to l_{1-} l_{2-}}$

$l_{2T} > l_{1T}$ $l_{1T} > l_{2T}$ $l_{2T} > l_{1T}$ $l_{1T} > l_{2T}$

$l_{2-} < \frac{1}{2}$ $l_{2-} > \frac{1}{2}$ $l_{2-} < \frac{1}{2}$ $l_{2-} > \frac{1}{2}$

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- This algorithm factorizes all overlapping singularities

Status

- The integrals take now the desired form

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- We checked that, for the case of the $q \rightarrow q\bar{q}qg$ contribution to the beam function, the weights $\mathcal{W}_j$ are finite in the limit of $x_i \rightarrow 0$, as required

- We are now ready to evaluate the integrals

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  2. Perfect agreement with analytic calculation for bubble graphs

  3. RG result for the complete NNLO soft function recovered: real and imaginary part

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    - We are now ready to evaluate all the divergent integrals

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