Remarks on transient modes in relativistic hydrodynamics

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Abstract

The successful application of relativistic hydrodynamics to the physics of quark-gluon plasma has inspired many theoretical studies of late time behaviour of systems initially far from equilibrium. There has been a growing appreciation of the role of non-hydrodynamic modes both at the microscopic level and in models formulated in the language of hydrodynamics. I will review this avenue of research including some very recent developments.

Introduction

- Post 2000: relativistic hydrodynamics becomes a key element of the theoretical picture of heavy-ion collisions
- 10 years ago: hydrodynamics describes small, long-lived perturbations of equilibrium
- Today: hydrodynamics is seen to work much more generally, but it is not quite clear why
- Key idea: causality requires non-hydrodynamic modes which act as a UV-regulator
- Important consequences: domain of applicability, attractors, systematic extensions of the hydrodynamic scheme

Origin of gapped modes: diffusion

Exact current conservation law

$$\partial_t \rho + \overrightarrow{\nabla} \cdot \overrightarrow{J} = 0$$

Constitutive relation in the gradient expansion

$$\overrightarrow{J} = D \overrightarrow{\nabla} \rho + \ldots$$

Keeping just the first order term gives the diffusion equation

$$\partial_t \rho + D \Delta \rho = 0$$

Modes (ungapped - “hydrodynamic”)

$$\rho = A e^{-i\omega t + ikz}, \quad \omega = -\, iDk^2$$

The “group velocity” is unbounded (and higher orders don’t help)

$$v = \frac{\partial \omega}{\partial k}$$

Introduce a relaxation time as a **regulator**:

$$\tau_R \partial_t \overrightarrow{J} + \overrightarrow{J} = D \overrightarrow{\nabla} \rho + \ldots$$

This generates gradients of all orders:

$$\overrightarrow{J} = D \overrightarrow{\nabla} \rho + \tau_R \partial_t \left(D \overrightarrow{\nabla} \rho \right) + \ldots$$

The dispersion relation becomes

$$-\tau_R \omega^2 + i\omega - Dk^2 = 0$$

Solutions

$$\omega_\pm = \frac{1}{2\tau_R}\left(-i \pm \sqrt{-1 + 4D\tau_R k^2}\right)$$

- Diffusive mode (old)

$$\omega_+ \sim -\, iDk^2 + \ldots$$

- Gapped/transient mode (new)

$$\omega_- \sim \frac{i}{\tau_R} + iDk^2 + \ldots$$

The relaxation time acts as a **regulator**: $v = \dfrac{\partial \omega}{\partial k} \sim \sqrt{D/\tau_R} \leq 1$

The "**pole collision**" sets the scale for **regulator independence**.

$$\omega_+ = \omega_- \iff 1/k_c = 2\sqrt{D\tau_R}$$

Linear perturbations

Perturbations of equilibrium lead to linear equations of the form

$$L\delta\Phi = 0$$

Solutions of the form of normal modes

$$\delta\Phi \sim e^{-i\omega t + ikz}$$

Dispersion relations are contained in

$$P(\omega, k) = 0$$

Each solution gives rise to a quasinormal mode.

In hydro theories this P is a polynomial; in microscopic theories this is in general not the case.

Relativistic Hydrodynamics

Conservation equation

$$\nabla_\alpha T^{\alpha\beta} = 0$$

Constitutive relations (needed a closed system of equations)

$$T^{\mu\nu} = \mathscr{E} u^\mu u^\nu + \mathscr{P}(\mathscr{E})(g^{\mu\nu} + u^\mu u^\nu) + \Pi^{\mu\nu}$$

First order: relativistic Navier-Stokes theory

$$\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu}, \quad \sigma^{\mu\nu} = \partial^\mu u^\nu + \cdots$$

Dispersion relations for linearised perturbations reveal purely hydrodynamic modes.

Sound channel dispersion relation

$$\omega = \frac{\eta}{Ts}k^2 \implies v = 2\frac{\eta}{Ts}k$$

so this theory is acausal and needs a "UV-completion" to avoid conflict with causality.

There are two known ways to do it:

- Mueller; Israel Stewart (& generalisations)
- Bemfica Disconzi Noronha; Kovtun

In both cases the dispersion relations show regularisation of group velocity due to a relaxation time parameter so they can be viewed as different UV-regularisations of relativistic Navier-Stokes hydrodynamics.

Mueller Israel Stewart theory

The same trick as before (other choices possible!):

$$\left(\tau_\pi \mathscr{D} + 1\right) \Pi^{\mu\nu} = -\,\eta\sigma^{\mu\nu} + \ldots$$

implies contributions to all orders in gradients

$$\Pi^{\mu\nu} = -\,\eta\sigma^{\mu\nu} + \tau_\pi \mathscr{D}(\eta\sigma^{\mu\nu}) + \ldots$$

and introduces purely-damped non-hydrodynamic modes.

$$\omega_{\mathrm{H}}^{(\pm)} = \pm \frac{k}{\sqrt{3}} - \frac{2i}{3\,T}\frac{\eta}{s}k^2 + \ldots \qquad\qquad \omega_{\mathrm{NH}} = -\,i\left(\frac{1}{\tau_\pi} - \frac{4}{3\,T}\frac{\eta}{s}k^2\right) + \ldots$$

Group velocity

$$v = \frac{1}{\sqrt{3}}\sqrt{1 + 4\,\frac{\eta/s}{T\tau_\pi}} < 1 \iff T\tau_\pi > 2\eta/s$$

BDNK causal first-order theory

The most general form of the EM tensor at first order is

$$T^{\mu\nu} = \mathscr{E} u^\mu u^\nu + \mathscr{P} \Delta^{\mu\nu} + (\mathscr{Q}^\mu u^\nu + \mathscr{Q}^\nu u^\mu) + \mathscr{T}^{\mu\nu}$$

with constitutive relations

$$\mathscr{E} = \epsilon + \varepsilon_1 \dot{T}/T + \varepsilon_2 \partial_\lambda u^\lambda$$

$$\mathscr{P} = p + \pi_1 \dot{T}/T + \pi_2 \partial_\lambda u^\lambda$$

$$\mathscr{Q}^\mu = \theta_1 \dot{u}^\mu + \theta_2 / T \, \Delta^{\mu\lambda} \partial_\lambda T$$

$$\mathscr{T}^{\mu\nu} = -\eta \sigma^{\mu\nu}$$

Usually one uses freedom to redefine the hydrodynamic fields by gradient terms to impose the Landau-frame conditions which eliminate the longitudinal contributions to the first order EMT.

Kovtun (following earlier work by Bemfica, Disconzi and Noronha) finds that if we refrain from imposing a frame choice then the resulting first order hydro equations are causal!

This does not mean that the theory contains hydro modes alone. Indeed, linearisation around equilibrium in the shear channel leads to the branch of the spectral curve

$$-\theta_1\omega^2 - iTs\,\omega - \eta k^2 = 0$$

which is of the same form as in MIS theory and leads to a gapped mode with frequency

$$\omega_{\mathrm{NH}} = -\,i\left(\frac{Ts}{\theta_1} + \frac{\eta}{Ts}\,k^2\right) + \dots$$

The story is similar in the sound channel.

Late time behaviour in MIS hydro

The shear channel dispersion relation (after some rescaling) is

$$\omega_{\pm} = -i \pm \sqrt{k^2 - 1}$$

The Green's function of the corresponding linear problem solves

$$(\partial_t^2 + 2\partial_t - \partial_x^2) G(t,x) = \delta(t)\delta(x)$$

It can be calculated exactly and satisfies causality constraints.

The non-hydrodynamical mode contribution is essential

$$G(t,x) = \theta(t)\big(I_+(t,x) + I_-(t,x)\big)$$

$$I_{\pm}(t,x) = \frac{1}{2\pi}\int_0^{\infty} dk \, \frac{e^{i(kx - \omega_{\pm} t)}}{\omega_+ - \omega_-}$$

Using standard asymptotic methods one finds that the hydrodynamic pole leads to

$$G_s \equiv G(t,0) \sim \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{t}} \sum_{k=0}^{\infty} a_k t^{-k}$$

$$a_k = \frac{(-1)^k \Gamma(1/2 - k)}{2^k \Gamma(1/2 + k) \Gamma(1 + k)}$$

up to exponentially suppressed contributions.

This series is factorially divergent:

$$\frac{a_{k+1}}{a_k} \sim \frac{k}{2t}$$

The Borel transform can be done analytically

$$\mathscr{B}[\sqrt{t} G_s](\xi) = \sum_{n \geq 0} \frac{a_n}{\Gamma[n+1]} \xi^n = \frac{1}{\sqrt{2}\pi^{3/2}} K(\xi/2) \equiv B(\xi)$$

The elliptic function K has a cut on the real axis; the **branch point** is at the frequency of the non-hydrodynamic mode

Correspondingly, the Borel sum exhibits a complex ambiguity

$$\mathcal{S}_{\pm} = \frac{1}{\sqrt{t}} \int_{C_{\pm}} d\xi \, e^{-w\xi} \, B(\xi) = \frac{1}{2} e^{-t} \left(I_0(t) \pm \frac{i}{\pi} K_0(t) \right)$$

The exact answer is

$$I_+ = \frac{1}{2} e^{-t} \left(I_0(t) - \frac{i}{\pi} K_0(t) \right) = \mathcal{S}_-$$

$$I_- = \frac{1}{2} e^{-t} \left(\frac{i}{\pi} K_0(t) \right) = \frac{1}{2} \left(\mathcal{S}_+ - \mathcal{S}_- \right)$$

$$I_+ + I_- = \frac{1}{2} e^{-t} I_0(t) = \frac{1}{2} \left(\mathcal{S}_+ + \mathcal{S}_- \right)$$

The non-hydro mode contribution **cancels the imaginary part!**

Summary

- Ensuring causality in hydrodynamics seems to require the presence of non-hydrodynamic (gapped) modes

- The emergence of hydrodynamic behaviour is governed by the decay of non-hydrodynamic transients rather than local equilibration

- Late time asymptotic expansions of Green's functions carry information about the non-hydrodynamic modes

- It may be interesting and useful to formulate theories of hydrodynamics with specific non-hydrodynamic sectors

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