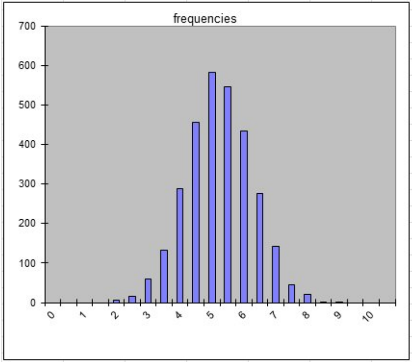
RANDOM VARIABLE and its CHARACTERISTICS

Visualizing Random Variable

suppose we have 3 000 numerical values. All these data follow a certain distribution – behave in a specific manner. In order to depict this behaviour we may construct a

Histogram



Histogram

the horizontal axis are the intervals ('bins') our values belong to. The range is (practically) from 2 to 9.5. And this range has been divided into 15 bins:

2-2.5	2.5-3	3-3.5	3.5-4	4-4.5	4.5-5	5-5.5	5.5-6	6-6.5	6.5-7	7-7.5	7.5-8	8-8.5	8.5-9	9-9.5
5	15	60	132	287	455	583	545	435	275	141	45	20	2	2

The second row shows how many values belong to a given interval: e.g. the third entry 60 shows that sixty values are greater than 3.0 and equal to or less than 3.5. In the given bin we have thus 60 out of 3000. **The probability that our RV** X has the values: $3.0 < x \leq 3.5$ is $60/3000 = 0.002$. The height of the vertical bar is the measure of this probability.

But for practical reasons we have to depict distributions with numbers rather than graphs. $\ddot{\frown}$

A FUNCTION OF A RANDOM VARIABLE: $Y = H(X)\ldots$

is also a random variable — so it also has — $F(y)$ a cumulative distribution with some *PARAMETERS* (which may be known from an experiment)

MATHEMATICAL EXPECTATION or
the MEAN VALUE OF A RANDOM VARIABLE

$$E(X) \;=\; \hat{x} = \begin{cases} \displaystyle\sum_{k=0}^{n} x_k \mathcal{P}(X = x_k) = \sum_{k=0}^{n} p_k x_k & \text{for a discrete RV} \\ \displaystyle\int_{-\infty}^{\infty} x f(x)\, dx & \text{for a continuous RV} \end{cases}$$

. . . every mathematical expectation is **a number**, so $E(X)$ is no longer something which may be called 'random'.
We use various conventions of notation: $E(X)$, $\hat{x}$, μ (the 'true' mean value for the given RV X) and m (the estimated mean value for the given X).
For a physicist (well, not only) $E(X)$ may be perceived as a "centre-of-mass" of the X, or . . .

weighted mean: $E(X) = \sum w_i x_i / \sum w_i$.

The weights are:
p_i's for discrete RV — $E(X) = \sum p_i x_i$
and $f(x)\,dx = \mathcal{P}(X \in [x, x + dx])$ for continuous RV
In the second case the sum is of course replaced by an integral.

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

MOMENTS OF RANDOM VARIABLES

LET OUR FUNCTION OF THE RANDOM VARIABLE V BE:

$$H(X) = (X - c)^l$$

(c –ANY NUMBER); ITS MATHEMATICAL EXPECTATION

$$E\{(X - c)^l\}$$

IS CALLED THE l-TH MOMENT OF THE RANDOM VARIABLE X WITH RESPECT TO c.

$$\alpha_l \stackrel{\text{def}}{=} E\{(X - c)^l\}$$

It is a logical to put $c = E(X)(\hat{x})$ — in this manner we obtain the so-called CENTRAL MOMENTS:

$$\mu_l = E\{(X - \hat{x})^l\}$$

MOMENTS OF RANDOM VARIABLES, cntd.

Let's consider the case of a continuous variable:

$$\begin{aligned}
\mu_0 &= \int_{-\infty}^{\infty} (x-\hat{x})^0 f(x)dx = 1 \\
\mu_1 &= \int_{-\infty}^{\infty} (x-\hat{x})^1 f(x)dx = 0 \\
\mu_2 &= \int_{-\infty}^{\infty} (x-\hat{x})^2 f(x)dx \stackrel{\text{def}}{=} VAR(X) = \sigma^2(X) = \text{VARIANCE} \\
\mu_3 &= \int_{-\infty}^{\infty} (x-\hat{x})^3 f(x)dx = \text{SKEWNESS} \\
\mu_4 &= \int_{-\infty}^{\infty} (x-\hat{x})^4 f(x)dx = \text{KURTOSIS}
\end{aligned}$$

MOMENTS OF RANDOM VARIABLES, cntd.

what is the meaning od those moments?

- VARIANCE — a measure of the spread (dispersion) (always > 0)
- SKEWNESS — a measure of asymmetry
- KURTOSIS— a measure of the spread as compared with a special type of distribution – normal distribution
- $$\sigma = \sqrt{VAR(X)} = \sigma(X) = \sigma_x$$

  — STANDARD MEAN DEVIATION OF A RANDOM VARIABLE X — N.B. it is expressed in the same UNITS as X!

$\sigma(X)$ ...

...may be regarded as a *natural unit* for measuring our Random Variable.

a short-cut formula for calculating the variance:

$$[X - E(X)]^2 = X^2 - 2E(X)X + [E(X)]^2 \qquad \star$$

but we have (X – a R.V.; a, b – constants)

$$E(aX + b) = aE(X) + b.$$

proof (for a discrete-type R.V)

$$\sum_i p_i x_i = \sum_i p_i(ax_i + b) = a\sum_i p_i x_i + b\sum_i p_i = aE(X) + b.$$

(Repeat this proof for the case of a continuous RV.)

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(Repeat this proof for the case of a continuous RV.)
Applying the E operator to the right member of the equation $\star$

$$E(X^2) - 2E(X)E(X) + E\{[E(X)]^2\} = E(X^2) - [E(X)]^2.$$

standardised MOMENTS OF RANDOM VARIABLES

One may prefer the so-called standardised parameters

$$\gamma_3 = \frac{\mu_3}{\sigma^3} \quad (= \gamma)$$
$$\gamma_4 = \frac{\mu_4}{\sigma^4} - 3 = \frac{\mu_4}{\mu_2^2} - 3$$

$$\gamma_3 > 0 \rightarrow E(X) - Mo > 0$$

$$\gamma_4 > 0 \rightarrow \text{the distribution is ,,slimmer'' than the Normal distribution}$$

$$Z = \frac{X - \hat{x}}{\sigma}$$

$\hat{x}$ — is a "natural" zero (origin)
σ — is a "natural" unit

Let X be a RV with $E(X) = \hat{x}$ and $VAR(X) = \sigma^2$. Then, for d being a number:

$$\mathcal{P}(|X - \hat{x}| \geq d) \leq \frac{\sigma^2}{d^2}, \quad \text{or}$$

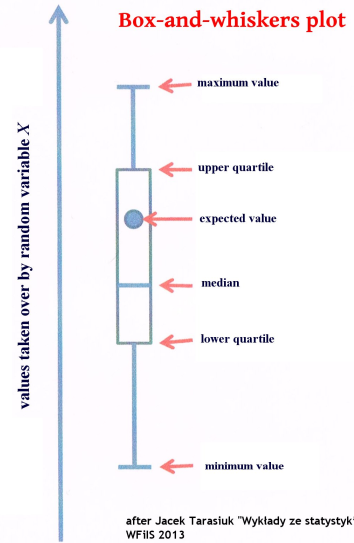
$$\text{putting: } d = k \cdot \sigma \quad \text{we get } \boxed{\mathcal{P}(|X - \hat{x}| \geq k \cdot \sigma) \leq \frac{1}{k^2}.}$$

This is CHEBYSHEV INEQUALITY – a rather crude estimate of the dispersion of our X around $E(X)$.

- quantile:
  A QUANTILE *q(f)* or x_f, is a value of x for which a specified fraction, f, of the X values is less than or equal to x_f:

  (1) $$F(x_f) = \mathcal{P}(X \leq x_f) \geq f$$
  (2) $$1 - F(x_f) = \mathcal{P}(X > x_f) \leq 1 - f$$

  (for a continuous RV we have the " $\geq$" or "$\leq$" sign)
  QUANTILE for $f = 0.5$ (50%) is called *median*; for $f = 0.25$ (25%) we have the first (lower) quartile, and for $f = 0.75$ (75%) we have the fourth (upper) quartile
- MODE (modal value — $\mathrm{Mo}(X)$)
  is a value x, for which: $\frac{df}{dx} = 0$ and $\frac{d^2 f}{dx^2}| < 0$
  — (local maximum of the probability density function)
- RANGE: $x_{max} - x_{min}$



after Jacek Tarasiuk "Wykłady ze statystyki inżynierskiej"
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DESCRIPTIVE STATISTICS