PROBABILITY – BASIC QUESTIONS

1. Four cards are to be dealt successively, at random and without replacement, from an ordinary deck of playing cards. Find the probability of receiving a spade, a heart, a diamond, and a club, in that order.

   We apply *the multiplication rule*:

   $$\mathcal{P}(S,H,D,C) = \mathcal{P}(S)\mathcal{P}(H|S) \times \mathcal{P}(D|S,H) \times \mathcal{P}(C|S,H,D)$$

   We have

   $$\mathcal{P}(S) = \frac{\text{number of spades}}{\text{number of cards in the deck}} = \frac{13}{52}.$$
   $$\mathcal{P}(H|S) = \frac{\text{number of hearts}}{\text{number of cards after a spade is drawn}} = \frac{13}{51}.$$
   $$\mathcal{P}(D|S,H) = \frac{\text{number of diamonds}}{\text{number of cards after a spade and a heart are drawn}} = \frac{13}{50}.$$
   $$\mathcal{P}(C|S,H,D) = \frac{\text{number of clubs}}{\text{number of cards after a spade, a heart and a diamond are drawn}} = \frac{13}{49}.$$

   Thus

   $$\mathcal{P}(S,H,D,C) = \frac{13}{52} \times \frac{13}{51} \times \frac{13}{50} \times \frac{13}{49} = \ldots = 0.0044$$

2. (Bayes scheme). Twenty percent of the employees of a company are college graduates. Of, these 75% are in supervisory positions. Of those who did not attend college, 20% are in supervisory positions. What is probability that a randomly selected supervisor is a college graduate?

   the notation:

   $\mathcal{P}(G)$ – probability a person IS college graduate —- 0.2;

   $\mathcal{P}(\bar{G})$ – probability a person IS NOT college graduate — 0.8;

   $\mathcal{P}(S|G)$ – probability a college graduate is a supervisor —- 0.75;

   $\mathcal{P}(S|\bar{G})$ – probability a NOT-college graduate is a supervisor —- 0.2;

   Finally, $\mathcal{P}(G|S)$ – probability that a randomly selected supervisor IS a college graduate — ???.

   the basic formula

   $$\mathcal{P}(S \cap G) = \mathcal{P}(G|S) \times \mathcal{P}(S) = \mathcal{P}(S|G) \times \mathcal{P}(G)$$

   hence

   $$\mathcal{P}(G|S) = \frac{\mathcal{P}(S|G) \times \mathcal{P}(G)}{\mathcal{P}(S)}$$

   But $\mathcal{P}(S) = \mathcal{P}(G)\mathcal{P}(S|G) + \mathcal{P}(\bar{G})\mathcal{P}(S|\bar{G}) = 0.2 \times 0.75 + 0.8 \times 0.2 = 0.31$

   Hence

   $$\mathcal{P}(G|S) = \frac{0.75 \times 0.2}{0.31} = 0.15/0.31 \approx 0.5$$

3. If 4 different balls are placed at random in 3 different boxes, find the probability that no box is empty. Assume that any box can hold up to 4 balls.

   There are 3 ways of placing each ball in a box. Thus the event space is

$$\Omega = 3^4.$$

   Now, the placement without any box empty follows the scheme: two balls in one of the boxes (we have three to choose from) and each of remaining balls in separate box – we have two boxes for the first ball; just one for the second). The number of such arrangements is

$$\left[\binom{4}{2} \times 3\right] \times 2 \times 1 = 36$$

   The looked-for $\mathcal{P}$ is the ratio $36/81 = 4/9$.

4. From an ordinary of playing cards, cards are drawn at random and without replacement. Compute the probability that the third spade appears on the sixth draw.

$$\mathcal{P}(\text{3rd spade in the 6th draw}) = \mathcal{P}(S|\text{2 spades in five first draws})\times\mathcal{P}(\text{2 spades in five first draws}).$$

$$\mathcal{P}(\text{2 spades in five first draws}) = \frac{\binom{13}{2} \times \binom{39}{3}}{\binom{52}{5}}.$$

$$\mathcal{P}(S|\text{2 spades in five first draws}) = 11/47.$$

   Thus our $\mathcal{P}$is equal to

$$\frac{11}{47} \times \frac{\binom{13}{2} \times \binom{39}{3}}{\binom{52}{5}} \approx 0.274.$$

5. A box contains two balls, each of which may be either red or white. A ball is drawn and found to be red. Find the probability that the other is red.

   Hint: To start with, the box could contain 0, 1 or 2 red balls. BEFORE any ball is drawn it seems reasonable to assume that any of these three *hypotheses* are equally likely to be true:

$$\mathcal{P}(H0) = \mathcal{P}(H1) = \mathcal{P}(H2) = 1/3.$$

$$\mathcal{P}(\text{red is drawn}) = \mathcal{P}(R|H0)\mathcal{P}(H0)+\mathcal{P}(R|H1)\mathcal{P}(H1)+\mathcal{P}(R|H2)\mathcal{P}(H2) = 1/3(0+1/2+1) = 1/2.$$

   The Bayes formula gives

$$\begin{aligned}\mathcal{P}(H2 \text{ true and Red is drawn}) &= \mathcal{P}(H2 \text{ true}|\text{ Red is drawn}) \times \mathcal{P}(\text{Red is drawn}) \qquad (1)\\ &= \mathcal{P}(\text{Red drawn}|H2 \text{ true}) \times \mathcal{P}(H2 \text{ true}). \qquad (2)\end{aligned}$$

   Equating the right members of (1) and (2) we have

$$\mathcal{P}(H2 \text{ true}|\text{ Red is drawn}) = \frac{\mathcal{P}(\text{Red drawn}|H2 \text{ true})\mathcal{P}(H2 \text{ true})}{\mathcal{P}(\text{Red is drawn})} = \frac{1 \times 1/3}{1/2} = 2/3$$

6. A hand of five cards is to be dealt at random and without replacement from an ordinary deck of cards. Find the conditional probability of an ALL spade hand given that there will be at least 4 spades in the hand.

   C_1 – the event that there are at least four spades;

   C_2 – the event that there are five spades;

   $C_1 \cap C_2$ – intersection of these two events; since C_1 is contained in C_2 we have $C_1 \cap C_2 = C_2$

$$\mathcal{P}(C_2|C_1) = \frac{\mathcal{P}(C_1 \cap C_2)}{\mathcal{P}(C_1)} = \frac{\mathcal{P}(C_2)}{\mathcal{P}(C_1)} = \dots$$

$$\mathcal{P}(C_2) = \frac{\binom{13}{5}}{\binom{52}{5}}$$

$$\mathcal{P}(C_1) = \frac{\binom{13}{5} + \binom{13}{4}\binom{39}{1}}{\binom{52}{5}}$$

   Thus

$$\dots = \frac{\binom{13}{5}}{\binom{13}{5} + \binom{13}{4}\binom{39}{1}} = 0.044.$$

7. A bowl contains eight chips. Three of them are red; the remaining five are blue. Two chips are drawn at random and without replacement. What is the $\mathcal{P}$ that the first is red and the second – blue?

$$\mathcal{P}(\text{first Red}) = \frac{\#\text{of red chips}}{\text{total } \# \text{ of chips}} = 3/8.$$

$$\mathcal{P}(\text{blue in the 2nd draw if red in the first})\frac{\#\text{of blue chips}}{\text{total } \# \text{ of chips after the 1st draw}} = \frac{5}{8-1}.$$

$$\begin{aligned}\mathcal{P}(\text{red in 1st AND blue in 2nd}) &= \mathcal{P}(\text{first Red}) \times \mathcal{P}(\text{blue in the 2nd draw if red in the first})\\ &= \frac{3}{8} \times \frac{5}{7} = \frac{15}{56}.\end{aligned}$$

8. In a college 30% of the men and 20% of the women are studying math. Further, 45% of the students are women. What is $\mathcal{P}$ that a math student selected at random is a woman?

   Notation:

   $\mathcal{P}(G)$ – prob of a student being a girl – 0.45;

   $\mathcal{P}(B)$ – prob of a student being a boy – 0.55;

   $\mathcal{P}(M|B)$ – prob of a boy-student studying math – 0.3;

   $\mathcal{P}(M|G)$ – prob of a girl-student studying math – 0.2;

   $\mathcal{P}(G|M)$ – prob of a math student being a girl – ??.

   The usual Bayes scheme gives

$$\mathcal{P}(G|M) = \frac{\mathcal{P}(G)\mathcal{P}(M|G)}{\mathcal{P}(G)\mathcal{P}(M|G) + \mathcal{P}(B)\mathcal{P}(M|B)} = \frac{0.45 \times 0.2}{0.45 \times 0.2 + 0.55 \times 0.3} = 6/17.$$

9. There is a box containing 5 White balls, 4 Black balls, and 7 Red ones. If two balls are drawn at a time from the box and neither is replaced, find the probability that

   a) both balls will be white;

   b) the first ball will be white, an the second red;

   c) if a third ball is drawn, find the probability that the three balls will be drawn in the order White, Black, Red.

$$a)\quad \frac{5}{16}\times\frac{4}{15}=1/12.$$
$$b)\quad \frac{5}{16}\times\frac{7}{15}=7/48.$$
$$c)\quad \frac{5}{16}\times\frac{4}{15}\times\frac{7}{14}=1/24.$$

10. Suppose a die has been loaded so that '1' face lands uppermost 3 times as often as any other face while all the other faces occur equally often. What is the probability of '2' on a single toss? What is the probability of '1'?

    b) event space $5+3=8$; hence $\mathcal{P}=3/8$;

    a)$\mathcal{P}=1/8$.

11. A card is drawn at random from a deck of cards. Find the $\mathcal{P}$ that AT LEAST one of the following three events will occur:

    Event A: a heart is drawn (13);

    Event B: a card which is not a face card is drawn (40);

    Event C: the number of spots (if any) of the drawn card is divisible by three (12 cards).

    caution: 10 cards are hearts AND non-face; 3 hearts divisible by 3; 12 cards non-face AND divisible by 3

    BUT a straight subtracting is WRONG: there are 3 cards that have all properties. Therefore

$$\mathcal{P}(A\cup B\cup C)=\frac{40+13+12-103-12+3}{52}=43/52.$$
$$\mathcal{P}(A)=13/52;\quad \mathcal{P}(B)=40/52;\quad \mathcal{P}(C)=12/52;$$
$$\mathcal{P}(AB)=10/52;\quad \mathcal{P}(AC)=3/52;\quad \mathcal{P}(BC)=12/52.$$

thus we proved

$$\mathcal{P}(A\cup B\cup C)=\mathcal{P}(A)+\mathcal{P}(B)+\mathcal{P}(C)-\mathcal{P}(AB)-\mathcal{P}(AC)-\mathcal{P}(BC)+\mathcal{P}(ABC).$$