CHEBYSHEV INEQUALITY
CENTRAL LIMIT THEOREM
and
The Law of Large Numbers

Let X be a RV with $E(X) = \mu$ and $VAR(X) = \sigma^2$. Then, for d being a number:

$$\begin{aligned} \mathcal{P}(|X-\mu| \geq d) &\leq \frac{\sigma^2}{d^2}, \quad \text{or} \\ \mathcal{P}(|X-\mu| \geq k \cdot \sigma) &\leq \frac{1}{k^2}. \end{aligned}$$

Verification:

$$\begin{aligned} \sigma^2 &= \sum_{all \quad x} (x-\mu)^2 \, \mathcal{P}(x) \geq \sum_{(x:\ |x-\mu| \geq d)} (x-\mu)^2 \, \mathcal{P}(x) \\ &\geq \sum_{(x:\ |x-\mu| \geq d)} d^2 \, \mathcal{P}(x) = d^2 \mathcal{P}(|X-\mu| \geq d) \quad q.e.d. \end{aligned}$$

GIVEN :
a sequence of n INDEPENDENT RANDOM VARIABLES X_i.
These RVs follow (unknown but of the same type) distributions with parameters:
E$\{X_i\} = \mu_i$; VAR$\{X_i\} = \sigma_i^2$.
THEN: The RV:

$$S = \sum_{i=1}^{n} X_i \quad \text{has} \quad E\{S\} = \sum_i \mu_i \qquad VAR(S) = \sum_i \sigma_i^2$$

and for $n \to \infty$ we have:

$$\frac{S - \sum_i^n \mu_i}{\sqrt{\sum_i^n \sigma_i^2}} \to N(0,1)$$

CENTRAL LIMIT THEOREM ...

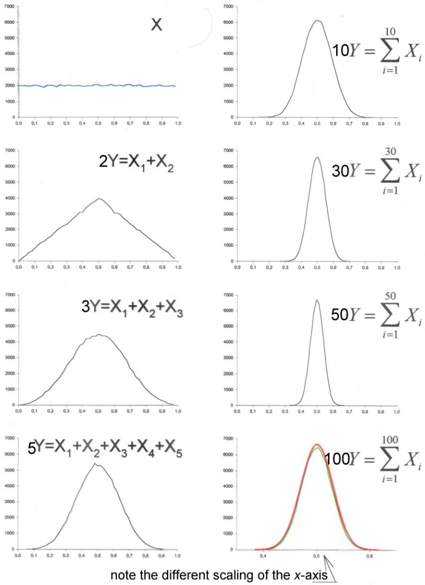
If all the X_i variables are 'the same':

$$\mu_i \equiv \mu; \quad \sigma_i \equiv \sigma$$

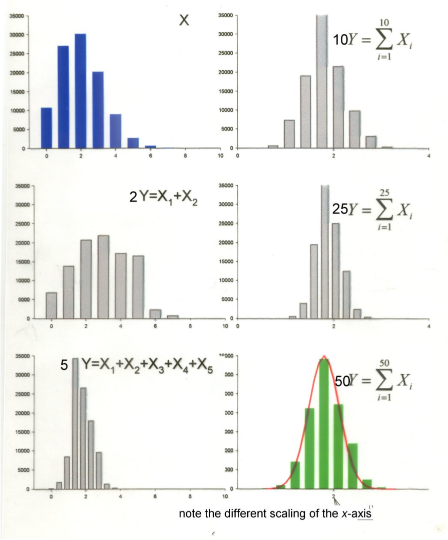
the RV $S = \sum_i^n X_i$ has the expected value $E(S) = n\mu$, and $VAR(S) = \sum_i \sigma_i^2 = n\sigma^2$ so we have:

$$\frac{S - n\mu}{\sqrt{n\sigma^2}} = \frac{\frac{S}{n} - \mu}{\frac{\sigma}{\sqrt{n}}} \to N(0,1)$$

IF WE ARE SAMPLING FROM A POPULATION WITH UNKNOWN DISTRIBUTION, EITHER FINITE OR INFINITE, THE SAMPLING DISTRIBUTION OF $\bar{X}$ WILL BE APPROXIMATELY NORMAL WITH MEAN μ AND VARIANCE σ^2/n PROVIDED THAT THE SAMPLE SIZE IS LARGE.


X
$10Y = \sum_{i=1}^{10} X_i$
2Y=X1+X2
$30Y = \sum_{i=1}^{30} X_i$
3Y=X1+X2+X3
$50Y = \sum_{i=1}^{50} X_i$
5Y=X1+X2+X3+X4+X5
$100Y = \sum_{i=1}^{100} X_i$
note the different scaling of the x-axis

CENTRAL LIMIT THEOREM . . .



The central limit theorem can be interpreted as follows: *for the sample size* $\to \infty$ *the arithmetic average* $\bar{X}_n$ *tends more and more closely to the expected value* $E(X) = \mu$. Or we can state:
Let $X_1, \ldots, X_n$ denote a sequence of independent Rvs with $E(X_i) = \mu$ and $VAR(X_i) = \sigma^2$. Then for every $d > 0$:

$$\boxed{\lim_{n\to\infty} P(|\bar{X}_n - \mu| > d) = 0.}$$

The proof follows immediately from the Chebyshev inequality: $VAR(X) = \sigma^2$ then $VAR(\bar{X}_n) = \sigma^2/n$. Thus

$$P(|\bar{X}_n - \mu| > d) \leq \frac{\sigma^2/n}{d^2}$$

and

$$\lim_{n\to\infty} P(|\bar{X}_n - \mu| > d) \leq \frac{\sigma^2/n}{d^2} = 0.$$